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# Caputo Sense Fractional Order Derivative Model of Cholera

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**Abstract:** A deterministic mathematical cholera model is formulated using ordinary differential equations. The formulated system of equations was then transformed into fractional derivative of Caputo sense, with order $\lambda$ that ranges between 0 and 1. The transformed equations were displayed in Caputo sense fractional order derivative using the fractional derivative operator. These equations were then interpreted and the numerical Adams-Bashforth-Moulton kind of predictor-corrector method was used on maple 18 software to obtain the model's outcome. Dynamics of cholera disease controls, comprising treatment, hygiene consciousness and vaccine were analyzed and the results were produced in graphs. The graphs show the dynamics of the susceptible, effects of vaccine on the susceptible and the rate of cholera infection. After studying and interpretation of the graphs, the result show that lower fractional order values in the range 0.25 to 0.5 gives lower values of susceptible and vaccinated individuals but gives higher number of infected individuals. To test efficiency of the obtained result, we compared it with the integer order derivative result, and found that the fractional order results gave a better and efficient, portray of the successful useable controls. Caputo sense fractional order derivative using Adams-Bashforth-Moulton kind of predictor-corrector numerical method, guaranteed getting result similar to Runge-Kutta fourth-order numerical method.

**Keywords:** Hygiene, Fractional Order, Numerical, Infection

## 1. Introduction

Application of fractional calculus in many fields of study that include; viscoelasticity, aerodynamics, biophysics, engineering, chemistry, finance and even modelling is on increase in recent years. Applying integer order or classical differential equations in modelling nonlinear complex systems such as biological processes that possess multi-scale behaviours and varying structures will not be sufficient in showing the entire dynamics therein. This can be successfully achieved through application of fractional order differential equations. As stated by M. Ozair et al. [13], in biology, it is deduce that, the membranes of cells of biological organisms have non-integer order electrical conductance this make it relevant for their classification in groups where there modelling is in non-integer order. Fractional epidemic models are considered by M. Awais et al. [1] to be another strong tool to explore the dynamics of infectious diseases in a better way than the ordinary integer order models.

In recent years, application of fractional calculus by many researchers has enormous increase due to its better capabilities of representing real world problems than the regular calculus as stated in F. Evirgen and M. Yavuz [8]. Its use has advantage of being a better way of describing hereditary materials and memory terms in real world phenomena [5]. The fractional differential operators are not as common as local operators, they consider the entire history of all the functions involve into account. The memory effect characteristic thereby makes it highly suitable when dealing with phenomenon in physical context. Fractional derivative has a memory capability of tracing the spread of epidemic from beginning to the infected individual [4]. Introduction of fractional order derivatives has greatly enhance nonlinear mathematical models, as stated by E. F. Doungmo Goufo [7], it is believe to generalize the entire ordinary differentiation and integration of non-integer and complex order. Models developed in many areas of science and engineering are observed to be

best explained by fractional differential equations [9].

Among the fractional order derivatives, Coputo sense is more suitable for application in initial value problem and those with boundary conditions as opined by I. Podlubny [15], the book further stated that initial conditions of fractional order differential equations take the same form as for integer order differential equations when applying Caputo sense, is recommended. K. Diethelm et al. [6] also preferred application of Caputo sense when dealing with concrete and physical real world applications with most data having well elaborated physical meaning that can be quantified, measured and qualified.

Preference of Caputo sense in this study is due to somewhat cumbersome and complications in the solutions of mathematical definitions and expressions in fractional order differential equations. Caputo sense of fractional order derivative, possess many interesting properties that are useful for modellers of real world problems as stated in N. R. O. Bastos [2], its temporal and spatial variables have different representation, hence this study prefer to use it. Vaccine model of cholera will be better-modelled using Caputo fractional derivative to capture the current and past stages of body immunity [11]. I. Bazhlekov and E. Bazhlekova [3] stated that modelling with time fractional derivative should be through the corresponding constitutive equation, since the integer order time derivative in a classical model comes from corresponding conservative law that a direct replacement of the derivative with fractional order derivative can lead to violation of the corresponding conservation law. A contrary finding was earlier found in K. M. Owolabi [12] who claimed that the pattern formation in integer order systems are similar to the fractional order cases. We compared the result of Caputo sense fractional order 1 with the result of Runge Kutta Fourth order.

Cholera is and has been an explosive disease that poses threat to unhygienic population especially where standard hygiene measures are not observe or breakdown of standard (especially) water supply infrastructures. It is a disease caused by bacteria called Vibrio cholerae, it dehydrates the patient and can cause death within short period of onset. Recently, cholera cases were reported in most states of Nigeria in the year 2021. Nigeria Centre for Disease Control (NCDC) reported that, as of 21st September 2021, a total of 73,055 suspected cholera cases that resulted to 2,407 deaths [10].

This paper contain SVICUTRB cholera model that divide the human population under study into seven compartments and one non-human compartment. Severity of the disease (cholera) with respect to time determine the rate of transfer between compartments. The rates of transfers are mathematically expressed into ordinary derivatives with respect to time, forming the model's system of ordinary differential equations which are converted to fractional differential equations. Necessity of vaccine in alerting body immunity is paramount especially during disease outbreak such as cholera, people with weak immunity need vaccine to complement their immunity, and hence vaccine compartment is included in the model.

## 2. Model Description

State variables or compartments of the model are:

$S(t)$ represent Susceptible Individuals; $V(t)$ the Vaccinated Individuals; $I(t)$ the Infected Individuals; $C(t)$ the Hygiene Conscious Individuals; $U(t)$ Untreated.

Vibrio cholerae Carriers; $T(t)$ Infected Individuals under Treatment; $R(t)$ the Recovered Individuals and $B(t)$ the Population of Vibrio cholerae Bacteria.

The brief description of the parameters is: the entire human population is denoted by $P_H$, $q$ is the rate of recruitment, $\delta$ is the per capital natural human death, $v_1$ is the rate of ingesting Vibrio cholerae bacteria from consumption of contaminated food and water, $v_2$ is the rate of transmission of Vibrio cholerae through human to human interaction. $\psi$ is the rate of vaccine, $\omega$ is the rate of waning out of the vaccine induced immunity, rate at which the vaccine reduce infection is $\sigma$, $\theta$ is the recovery rate of hygiene conscious individuals while $h$ is the hygiene consciousness of individuals. $\beta$ is the rate Vibrio cholerae decrease, $\eta$ is the rate at which infected individuals increase Vibrio cholerae bacteria in the environment. $f$ is the rate of freedom from Vibrio cholerae bacteria by the hygiene conscious individuals, $k$ is the concentration of cholera pathogens that yields 50% chance of individual developing cholera disease, $g_1$ is the progression of recovery of treated individuals to untreated Vibrio cholerae carries.

$g_2$ is the rate of interruption of treatment, $r_1$ is the recovery rate of untreated Vibrio cholerae carries, $r_2$ is the rate of recovery of treated cholera patients, $\epsilon$ is cholera induced death rate for individuals in $I$ compartment. $\mu$ is cholera disease induced death rate for individuals in $U$ compartment. $\alpha$ is disease induced death rate for individuals in $T$ compartment, $m$ is the rate of movement of the infected individuals, $\tau$ is the probability of movement of early infected $mI$ to $T$ compartment. The linear incidence of the model representing the human-to-human transmission and the nonlinear incidence representing the logistic response to the increase in the cholera pathogen denoted by $v_1 \frac{B}{k+B}$ where $\frac{B}{k+B}$ is represented by $\varphi$.

Figure 1 gives the schematic representation of the model, from it we obtain the model's equations.
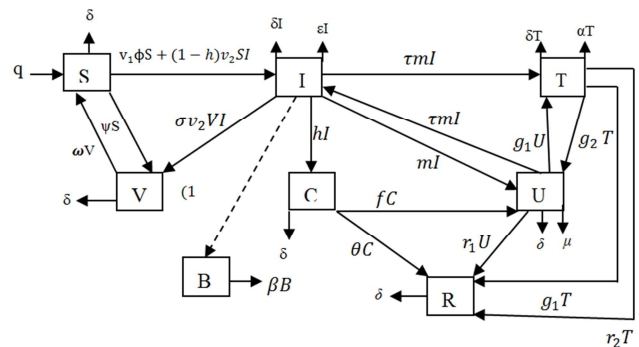


*Figure 1. Schematic diagram of the SVICUTRB model.*

Obtained from the diagram is the following system of integer order differential equations.

# 3. Main Result

D is mostly the notation used for fractional order derivative, there are many notations in literature for the D operator, R. K. Shukla and P. Sapra [16] suggested $_aD_t^\alpha f(t)$ as fractional derivative notation, where $t$ is the upper limit, $a$ is the lower limit and $\alpha$ is the order of the fractional derivative. As in E. C. Oliveira and J. A. T. Machado [14], forms and manners of the notations are:

$$D_{a+}^\alpha f(t) = (D_{a+}^\alpha f)(t) = {}_aD_t^\alpha f(t) = {}_aI_t^{-\alpha} f(t) = D_{t-a}^\alpha f(t) = \frac{d^\alpha f(t)}{d(t-a)^n}$$

## 3.1. Fractional Order (λ) Formulation of the Model

The formulated SVICUTRB model describe by system of ordinary differential equations (1), is now written in the Caputo sense fractional derivative of order $\lambda$, with a range $0 \le \lambda \le 1$ and the Caputo fractional order derivative operator $^CD_t^\lambda$.

$$\left.\begin{aligned}
\frac{dS}{dt} &= q - v_1\phi S - (1-h)v_2 SI + \omega V(\psi + \delta)S \\
\frac{dV}{dt} &= \psi S + \sigma v_2 VI - (\omega + \delta)V \\
\frac{dI}{dt} &= v_1\phi S + (1-h)v_2 SI - \sigma v_2 VI - (m + h + \delta + \epsilon)I \\
\frac{dC}{dt} &= hI - (f + \theta + \delta)C \\
\frac{dU}{dt} &= mI + fC + g_2 T - \tau mI - (g_1 + r_1 + \delta + \mu)U \\
\frac{dT}{dt} &= \tau mI + g_1 U - (g_1 + g_2 + r_2 + \delta + \alpha)T \\
\frac{dR}{dt} &= \theta C + r_1 U + (g_1 + r_2)T - \delta R \\
\frac{dB}{dt} &= (1-h)\eta - \beta B
\end{aligned}\right\} \tag{1}$$

System of the Caputo fractional order $\lambda$ derivative is:

$$\left.\begin{aligned}
^CD_t^\lambda &= q - v_1\phi S - (1-h)v_2 SI + \omega V(\psi + \delta)S \\
^CD_t^\lambda &= \psi S + \sigma v_2 VI - (\omega + \delta)V \\
^CD_t^\lambda &= v_1\phi S + (1-h)v_2 SI - \sigma v_2 VI - (m + h + \delta + \epsilon)I \\
^CD_t^\lambda &= hI - (f + \theta + \delta)C \\
^CD_t^\lambda &= mI + fC + g_2 T - \tau mI - (g_1 + r_1 + \delta + \mu)U \\
^CD_t^\lambda &= \tau mI + g_1 U - (g_1 + g_2 + r_2 + \delta + \alpha)T \\
^CD_t^\lambda &= \theta C + r_1 U + (g_1 + r_2)T - \delta R \\
^CD_t^\lambda &= (1-h)\eta - \beta B
\end{aligned}\right\} \tag{2}$$

Initial conditions of the system of equations (2) are:

$$\left.\begin{aligned}
S(0) = E_1, V(0) = E_2, I(0) = E_3, C(0) = E_4, \\
U(0) = E_5, T(0) = E_6, R(0) = E_7, B(0) = E_8
\end{aligned}\right\} \tag{3}$$

Caputo fractional derivative of order $\lambda$ of the function $f(t)$ is define in R. K. Shukla and P. Sapra [16], as:
$_a^C D_t^\lambda f(t) = \frac{1}{\Gamma(n-\lambda)}\int_a^t (t-u)^{n-\lambda-1} f^{(n)}(u)du, t \ge 0, (n-1) \le \lambda < n$ where n is an integer and $\lambda$ is real number. Taking the Caputo fractional order initial value problem $_a^C D_t^\lambda f(t) = g(t, f(t)), 0 \le t \le T, f^{(k)}(0) = f_0^{(k)}$, with $k = 0, 1, ..., n-1$ and $\lambda \in (n-1, n)$, where $g$ is a non-linear function and n is a positive integer. It can be transformed to Volterra integral equation thus:

$$f(t) = \sum_{k=0}^{n-1} f_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\lambda)}\int_a^t (t-u)^{n-\lambda-1} f^{(n)}(u)du$$

discretizing T with uniform grid $\{t_1 = ih : i = 0, 1, ..., N\}$ and $h := \frac{T}{N}$ with $f_h(t) \approx f(t)$.

The Adams-Bashforth-Moulton kind of predictor-corrector formula as given in K. Diethelm, et al. [6] is:

$$f_h(t_{i+1}) = \sum_{k=0}^{n-1} f_0^{(k)} \frac{t_{i+1}^k}{k!} + \frac{h^\lambda}{\Gamma(\lambda+2)} g(t_{i+1}, f_h^P(t_{i+1})) + \frac{h^\lambda}{\Gamma(\lambda+2)}\sum_{j=0}^i A_{j,i+1} g(t_j, f_h(t_j))$$

where

$$A_{j,i+1} = \begin{cases} (i)^{\lambda+1} - (i-\lambda)(i+\lambda)^\lambda \text{ if } j = 0 \\ (i-j+2)^{\lambda+1} + (i-j)^{\lambda+1} - 2(i-j+1)^{\lambda+1} \text{ if } 1 \le j \le i \\ 1 \text{ if } j = i+1 \end{cases}$$

And

$$f_h^P(t_{i+1}) = \sum_{k=0}^{n-1} \frac{t_{i+1}^k}{k!} f_0^{(k)} + \frac{1}{\Gamma(\lambda)} \sum_{j=0}^{i} B_{j,i+1} g(t_j, f_h(t_j))$$

Where $B_{j,i+1} = \frac{h^\lambda}{\lambda}((i+1-j)^\lambda - (i-j)^\lambda)$.

This Adams-Bashforth-Moulton kind of predictor-corrector formula is widely used as in the literature to obtain numerical solutions of fractional order systems of linear and non-linear equations. It is used in this paper to obtain the simulation of the model.

### 3.2. Numerical Result

Numerical simulations of the model with varied values of $\lambda$ where $(0 < \lambda \leq 1)$, as the fractional order of the derivatives of the model. Comparison between integer order and fractional order graphs are displayed. The chosen varied values of $\lambda$ are 0.25, 0.5, 0.75, 0.95 and 1, values of the parameters given in rates are: $q = 0:1$; $v_1 = 0:5$; $\phi = 0:000599$; $h = 0:6$; $v_2 = 0:03$; $\omega = 0:5$; $\psi = 0:3$;

$\delta = 0.0101$; $⊡ = 0:38$; $f = 0:6$; $g_1 = 0:08$; $g_2 = 0:2$; $\mu = 0:1$; $m = 0:2$; $\tau = 0:5$; $r_1 = 0:8$; $r_2 = 0:6$; $\sigma = 0:2$; $\epsilon = 0:1$; $\alpha = 0:08$; $\eta = 10$; $\beta = 0:5$.

State variables: $S = 0:033$; $V = 0:167$; $I = 0:002$; $C = 0:117$; $U = 0:006$; $T = 0:005$; $R = 0:0133$; $B = 0:0005$, reflecting memory property of fractional derivative.

Graphs in the figures show the dynamics of the compartments and comparisons between Runge-Kutta fourth order method simulation of integer order derivative and the fractional order simulation with Adams-Bashforth-Moulton kind of predictor-corrector method. Effect of rates of vaccine to susceptible individuals and the infection are in figures 2 and 4. In figure 2, the integer order Runge-Kutta fourth-order (RK4) method and Caputo fractional derivative method of order one tallied. Decreasing the order of the derivative $(\lambda)$ from one, result to a simultaneous decrease in the number of susceptible.
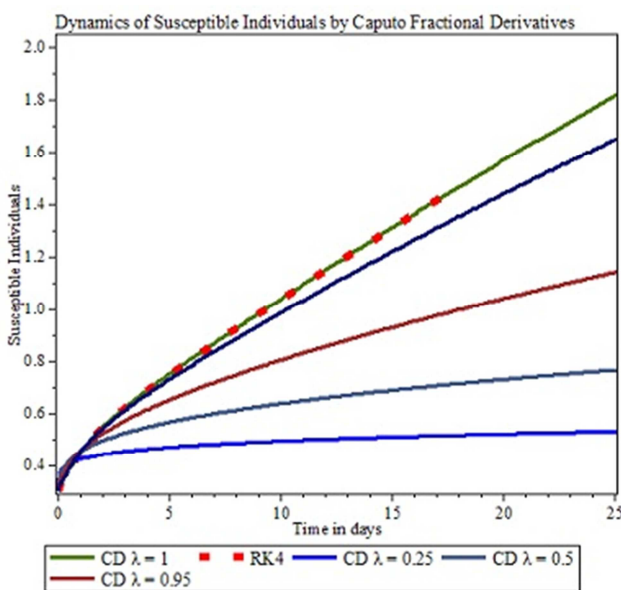


*Figure 3. Caputo sense fractional derivative of order $\lambda$ with $0 < \lambda \leq 1$, showing the dynamics of vaccinated individuals.*

In figure 3, order one Runge-Kutta fourth-order (RK4) method tallied with the result from Caputo fractional derivative method of order one, but reduction in the value of the order $\lambda$ shows decrease in rate of vaccinated individuals. Figure 4 show the dynamics of infected individuals in the model. Results from integer order Runge-Kutta fourth-order.

(RK4) method and the Caputo fractional derivative method of order one, show drastic drop of rate of infection, though RK4 result drop faster and steadier than the Caputo fractional derivative of order one. Reduction in the order $\lambda$ increases rate of infection. Figure 5 show the effect of rate of vaccine on the Susceptible as obtain from Caputo sense fractional derivative of order $\lambda$. Increase in rate of vaccine result to decrease in rate of susceptible.



*Figure 2. Graph of Caputo sense fractional derivative of order $\lambda$ with $0 < \lambda \leq 1$, showing the dynamics of susceptible individuals.*
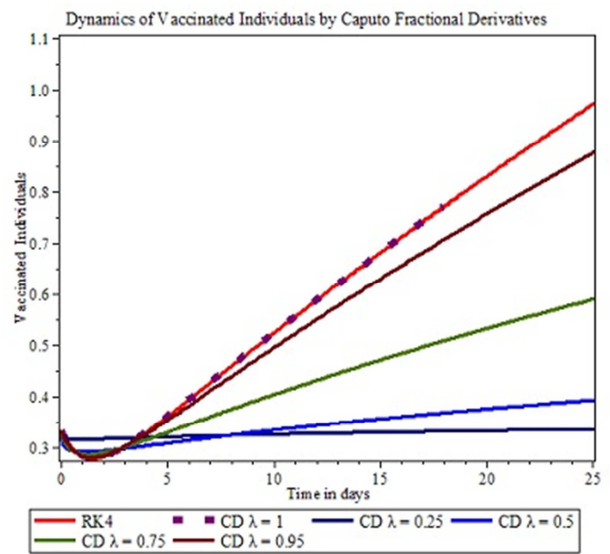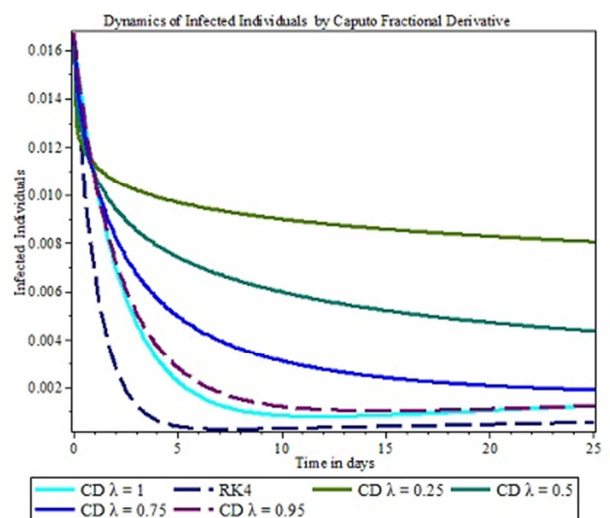


*Figure 4. Caputo sense fractional derivative of order $\lambda$ with $0 < \lambda \leq 1$, showing the dynamics of infected individuals.*
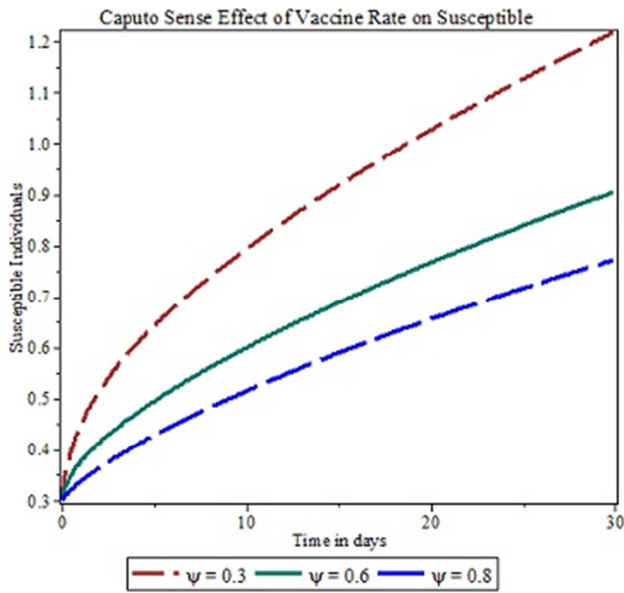
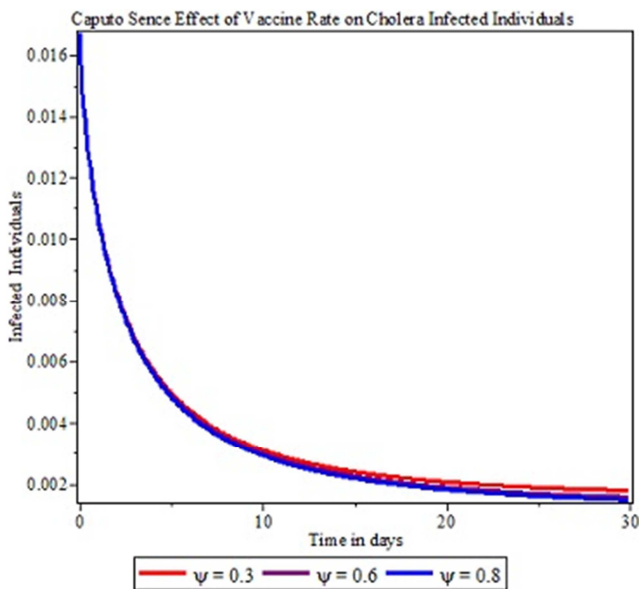**Figure 5.** *Effect of rate of vaccine on the Susceptible.*



**Figure 6.** *Effect of rate of vaccine on cholera infected individuals.*

Figure 6 show the effect of rate of vaccine on cholera infected individuals, obtained from Caputo sense fractional derivative of order $\lambda$. Increase in rate of vaccine does not change rate of infected individuals, hence vaccinating infected individuals do not cure infection. Comparison between integer order Runge-Kutta fourth-order (RK4) method and Caputo fractional derivative method of order one tallied in susceptible, vaccinated and infected compartmental graphs as shown in figure 6.

The results are interpreted as follows:

i.  As the order of $\lambda$ is reduced, a simultaneous reduction in susceptible individuals is attained.

ii.  Reduction in order of $\lambda$ shows decrease in rate of vaccinated individuals,

iii.  There is drastic drop in the rate of infection, RK4 result drop faster and steadier than the order one Caputo fractional derivative.

Hence, reduction in the value of order $\lambda$, show increase in rate of infection.

Effect of rate of vaccine on the Susceptible as obtained from Caputo sense fractional derivative of order $\lambda$, show that increase in rate of vaccine lead to decrease in rate of susceptible. But increase in rate of vaccine does not change rate of already infected individuals, hence vaccinating infected individuals does not reduce the infected individuals. Figures 5 and 6 show the dynamics of the compartments and comparisons between Runge-Kutta fourth-order method simulation of integer order derivative and the fractional order simulation using Adams-Bashforth-Moulton kind of predictor-corrector method.

Effect of rates of vaccine to susceptible individuals and the infection are shown in figures 2, 3 and 4. Effect of the vaccine in preventing cholera disease is analysed using fractional derivatives of Caputo senses. Numerical solutions of the model from four fractional orders (0.25, 0.5, 0.75, and 0.95) were obtained, number of susceptible was reduced as a result of the vaccination. Analyses of dynamics of cholera treatment, hygiene consciousness of individuals and cholera vaccination is performed.

## 4. Conclusion

We have obtained from this paper that: Usage of lower fractions $0.25 < \lambda < 0.5$ out of the $0 \leq \lambda \leq 1$ as order of the derivative gives lower number of susceptible and vaccinated individuals, but it increases the rate of infection. Generally the Caputo fractional derivative of integer order 1, using Adams-Bashforth-Moulton kind of predictor-corrector method gives a similar result with Runge-Kutta fourth-order method (RK4). Higher vaccine coverage drastically reduce the susceptible individuals but has no such effect on the infected individuals.

## Acknowledgements

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